Procedure for writing the contrast of the Main Effects and Interactions

The treatment sum of squares is now to be bifurcated into main effects and interactions. This can easily be done through contrast analysis. One has to define the set of contrasts for each of the main effects and interactions. Before describing the procedure of defining contrasts for main effects and interactions, we give some preliminaries. In general, let there be n-factors, say $F_1, F_2, ..., F_n$ and i^{th} factor has s_i levels, $i = 1,...,n$. The $\prod_{i=1}^{n} s_i$ treatment combinations in the lexico-graphic order are given by $\mathbf{a}_1 \times \mathbf{a}_2 \times ... \times \mathbf{a}_n$ where $\times$ denotes the symbolic direct product and $\mathbf{a}_i' = (0,1,...,s_{i-1})'$; $i = 1,2,...,n$. The total number of factorial effects (main effects and interactions) are $2^n - 1$. The set of main effects and interactions have a one-one correspondence with $\mathbf{\Omega}$, the set of all n-component non-null binary vectors. For example a typical p-factor interaction.
$F_{g_1}, F_{g_2}, ..., F_{g_p}$ $(1 \le g_1 \le g_2 \le ... \le g_p \le n,\ 1 \le p \le n)$ corresponds to the element $x = (x_1, ..., x_n)$ of $\mathbf{\Omega}$ such that $x_{g_1} = x_{g_2} = ... = x_{g_p} = 1$ and $x_u = 0$ for $u \ne g_1, g_2, ..., g_p$.

The treatment contrasts belonging to different interactions $F^x, x = (x_1, ..., x_n) \in \mathbf{\Omega}$ are given by

$\mathbf{P}^x\mathbf{t}$, where $\mathbf{P}^x = \mathbf{P}_1^{x_1} \otimes \mathbf{P}_2^{x_2} \otimes ... \otimes \mathbf{P}_n^{x_n}$

$$\text{where } \mathbf{P}_i^{x_i} = \begin{cases} \mathbf{P}_i & \text{if } x_i = 1 \\ \mathbf{1}'_{s_i} & \text{if } x_i = 0 \end{cases}$$

where $\mathbf{P}_i$ is a $(s_i - 1) \times s_i$ matrix of complete set of linearly independent contrasts of order s_i and $\mathbf{1}_{s_i}$ is a $s_i \times 1$ vector of ones. For example, if $s_i = 4$, then $\mathbf{P}_i = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix}$.

To illustrate the above consider the case of a $3 \times 4 \times 5$ factorial experiment. Let the three factors be represented by F_1 F_2 F_3. Then $\mathbf{\Omega}$, the set of non-null binary vectors is given by

$$\mathbf{\Omega} = \{100,\ 010,\ ,001,\ 110,\ 101,\ 011,\ 111\} .$$

Main effects and the interactions $F_1, F_2, F_3, F_1F_2, F_1F_3, F_2F_3$ and $F_1F_2F_3$ are represented by $F^{100}, F^{010}, F^{001}, F^{110}, F^{101}, F^{011}, F^{111}$

The $\mathbf{P}_i$ matrices for these three factors are

$$\mathbf{P}_1 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix};\ \mathbf{P}_2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix} \text{ and } \mathbf{P}_3 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 \\ 1 & 1 & 1 & -3 & 0 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}.$$

The coefficient matrices of the treatment contrasts for the above set of main effects and interactions are given by

Factorial Effect	Coefficient Matrix

$$F_1 \quad : \quad \mathbf{P}^{100} = \mathbf{P}_1 \otimes \mathbf{1}_4 \otimes \mathbf{1}_5$$
$$F_2 \quad : \quad \mathbf{P}^{010} = \mathbf{1}_3 \otimes \mathbf{P}_2 \otimes \mathbf{1}_5$$
$$F_3 \quad : \quad \mathbf{P}^{001} = \mathbf{1}_3 \otimes \mathbf{1}_4 \otimes \mathbf{P}_3$$
$$F_1F_2 \quad : \quad \mathbf{P}^{110} = \mathbf{P}_1 \otimes \mathbf{P}_2 \otimes \mathbf{1}_5$$
$$F_1F_3 \quad : \quad \mathbf{P}^{101} = \mathbf{P}_1 \otimes \mathbf{1}_4 \otimes \mathbf{P}_3$$
$$F_2F_3 \quad : \quad \mathbf{P}^{011} = \mathbf{1}_3 \otimes \mathbf{P}_2 \otimes \mathbf{P}_3$$
$$F_1F_2F_3 : \quad \mathbf{P}^{111} = \mathbf{P}_1 \otimes \mathbf{P}_2 \otimes \mathbf{P}_3 .$$

For sum of squares of these contrasts and testing of hypothesis, please see Contrast Analysis.
Now, the next thing in which experimenter is interested is the factorial effect means. Here the factorial effect means include the means for all levels of a single factor averaged over levels of all other factor means, means for all level combinations of two factors averaged over levels of all other factors and so on.
The means of all combinations of *p* (≥1) factors arranged over levels of all other factors can be obtained as follows:
Let $(j_1 j_2 \cdots j_p)$ represent a treatment combination of a *p*-factor combination, $j_i = 0,1,\ldots,s_i - 1$.
The mean corresponding to $\left(j_1 j_2,\ldots j_p\right)^{th}$ *p*-factor treatment combination is given by $M^{x}_{\left(j_1\ j_2\ \ldots\ j_p\right)}\mathbf{T}_A$ where $\mathbf{T}_A$ is the vector of $\prod_{i=1}^{n} s_i$ treatment combination means and

$$M^{\mathbf{x}}_{\left(j_1\ j_2\ \ldots\ j_p\right)} = M^{x_1}_{\left(j_1\ j_2\ \ldots\ j_p\right)} \otimes M^{x_2}_{\left(j_1\ j_2\ \ldots\ j_p\right)} \otimes \ldots \otimes M^{x_n}_{\left(j_1\ j_2\ \ldots\ j_p\right)}$$

with

$$\mathbf{M}^{x_i}_{\left(j_1\ j_2\ \ldots\ j_p\right)} = \mathbf{M}_{j_i} \text{ if } x_i \text{ is one among the p-factor}$$
$$= \mathbf{1}_{s_i} \text{ if } x_i \text{ is not one among the p-factor}$$

Here $\mathbf{M}_{j_i}$ is a $s_i \times 1$ vector with unity at the s_i^{th} position and remaining elements as zero.
If $\mathbf{M}^{x_i}_{\left(j'_1\ j'_2\ \ldots\ j'_p\right)}$ represent the coefficient matrix for obtaining the mean of $\left(j''_1\ j'_2 \ldots\ j'_p\right)^{th}$ combinations. Then

$$\mathbf{L}^{\mathbf{x}} = \mathbf{M}^{x}_{\left(j_1\ j_2\ \ldots\ j_p\right)} - \mathbf{M}^{x_i}_{\left(j'_1\ j'_2\ \ldots\ j'_p\right)}.$$

can be used as a coefficient matrix of the treatment effects for comparing the effects of $\left(j_1\ j_2\ \ldots\ j_p\right)^{th}$ and $\left(j'_1\ j'_2\ \ldots\ j'_p\right)^{th}$ treatment combinations. To be clearer, consider the example of $2\times3\times4$ factorial experiment with factors as A, B and C. Suppose that one is interested in obtaining the means of level combinations of factors A and B averaged over levels of factor C. The 6 factor combinations are 00, 01, 02, 10, 11, 12, 20, 21, 22. Then the mean for $(02)^{\text{th}}$ combination is given by $M_{(02)}\,\mathbf{T}_A$ where

$$M_{(02)} = (1\ 0) \otimes (0\ 0\ 1) \otimes (1\ 1\ 1\ 1)$$
$$= (0\ 0\ 1\ 0\ 0\ 0) \otimes (1\ 1\ 1\ 1)$$

$$= \left(\mathbf{0}'_8 \ \mathbf{1}'_4 \ \mathbf{0}'_{12}\right).$$

Further, if all the level combination of p-factors are appearing same number of times in each of the 'b' blocks, say a, then we can obtain the least significant difference for comparing the effect of p-factor combinations means averaged over levels of all other factors by

$$lsd = t_{\alpha, edf} \times \sqrt{\frac{2\,MSE}{ab}}\ .$$